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# Midterm Exam of Black Hole Physics

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This exam is take-home and you have five days to return the exam sheets. **Total mark: 100**.

**1)** Let $\xi_H$ be the Killing horizon generating Killing vector field of a given black hole solution which has other Killing vectors $\xi^{(i)}$. (**2+5=7 points**).

1-1) Show that

$$\nabla_\mu \xi_H^2 = -2\kappa(\xi_H)_\mu\,, \tag{1}$$

where $\kappa$ is surface gravity.

1-2) For which $\rho^{(i)}$'s and $\xi^{(i)}$'s the equation below is consistent:

$$\xi_H^\mu \nabla_\mu \xi_\nu^{(i)} = -\rho^{(i)}\xi_\nu^{(i)}\,. \tag{2}$$

**2) Null Gaussian Coordinates**. For a $d$-dimensional black hole Null Gaussian Coordinates (NGC) consists of coordinates $(v, r, x_i)$, $i = 1, 2, \cdots, d-2$ where
i) horizon is at $r = 0$;
ii) $\partial_v$ is the Killing horizon generating Killing vector field, which is timelike everywhere outside the horizon;
iii) $|\partial_r|^2 = 0\,, \partial_r \cdot \partial_v = 1$.

Write down Kerr black hole in Null Gaussian coordinates. (**10 points**).

**3) Kerr-Schild Coordinates**. Consider metrics of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\mathcal{H}\mathbf{k}_\mu\mathbf{k}_\nu \tag{3}$$

where $\eta_{\mu\nu}$ is Minkowski metric, $\mathcal{H}$ is a scalar function of spacetime and $\mathbf{k}_\mu$ is a null vector field. (**2+2+4+4+5+5=22 points**).

3-1) Show that if $\mathbf{k}_\mu$ is null w.r.t. $g_{\mu\nu}$ it is also null w.r.t. to $\eta_{\mu\nu}$.

3-2) Write down the inverse metric $g^{\mu\nu}$.

3-3) Study the geodesic structure of a metric in Kerr-Schild form in terms of $\mathcal{H}$ and $\mathbf{k}_\mu$.

3-4) What is the condition on $\mathcal{H}$ which makes $\mathbf{k}_\mu$ a Killing?

3-5) Write down the Reissner-Nordstrom metric in the Kerr-Schild form. What are $\mathcal{H}$ and $\mathbf{k}_\mu$ in this case? Can one write the gauge field $A_\mu$ for this solution in terms of $\mathbf{k}_\mu$?

3-6) Show that Kerr metric can be written in Kerr-Schild form. What are $\mathcal{H}$ and $\mathbf{k}_\mu$ in this case?

**4) Majumdar-Papapetrou (MP) Solutions.** Consider 4d Einstein-Maxwell theory and show that there is a family of **static** solutions of the form

$$\begin{aligned} ds^2 =& -F(x_i)^{-2}dt^2 + F(x_i)^2 dx_i dx_i, \quad i = 1, 2, 3, \\ A =& F(x_i)^{-1}dt. \end{aligned} \tag{4}$$

(**5+3+3+3=14 points**).

4-1) What is the condition on function $F(x_i)$?

4-2) Write down the most general solution in the MP class which is also spherically symmetric.

4-3) Write down the form of solution in the MP class which has translation symmetry along $x$-axis.

4-4) Write down the form of solution in the MP class which has $ISO(2)$ (isometries on 2d plane).

**5) Killing Horizons in flat and AdS space.** Killing horizons are codimension two hypersurfaces in spacetime generated by the orbits of a null Killing vector field $\xi$. In particular, Killing horizons are observer (coordinate) dependent and are not necessarily unique in a given spacetime. In this problem we explore this issue. (**2+2+5+2+2+5+2+5=25 points**).

5-1) Consider a 4d flat space in Cartesian coordinates $(t, x, y, z)$ and the vector field $\xi_1$

$$\xi_1 = t\partial_x + x\partial_t\,. \tag{5}$$

i) Show that $\xi_1$ is a Killing.
ii) Where is the Killing horizon generated by $\xi_1$?
iii) Is this Killing horizon a bifurcate horizon? What is the surface gravity associated with $\xi_1$?

5-2) For the same flat space consider the vector fields $\xi_2^\pm$

$$\xi_2^\pm = y\partial_t + (t \pm x)\partial_y \mp y\partial_x\,. \tag{6}$$

iv) Show that $\xi_2^\pm$ are Killing.
v) Where is the Killing horizon generated by $\xi_2^\pm$?
vi) Is this Killing horizon a bifurcate horizon? What is the surface gravity associated with $\xi_2^\pm$?

5-3) Consider an $AdS_4$ space time with metric

$$ds^2 = -(r^2 - k)dt^2 + \frac{dr^2}{r^2 - k} + r^2 d\Sigma_k^2\,, \tag{7}$$

where $d\Sigma_k^2$ denote the metric for a Euclidean 2d surface of constant curvature $k$ and $k = -1, 0, +1$.
vii) Show that $\partial_t$ is the Killing horizon generating Killing vector for $k = 0, 1$ cases.
viii) Compute surface gravity for $k = 0$ and $k = 1$ cases.

**6) More on ergo-region.** In the Kerr geometry ergo-sphere is where the Killing vector $\partial_t$ becomes null. (**3+4+3+5=15 points**).

6-1) Can we then conclude that ergo-sphere is a Killing horizon? Why?

6-2) Show that there are no static causal curves inside the ergo-region; they should rotate with the black hole.

6-3) Find the Killing vector field which is time-like everywhere outside the event horizon of the Kerr geometry.

6-4) Is (are) there Killing vector(s) which is (are) hypersurface orthogonal at the horizon?

**7) Maximum energy extraction.** For a Kerr black hole one can extract energy from the hole through Penrose process. Show that the maximum amount of extractable energy from a Kerr black hole is around 30% of its mass. (**7 points**).